M.A. / M.Sc. (Third Semester) Examination, 2013

AS-2227 Mathematics (Integration Theory - I)

Solution

Section A

1 (i) $m^*(A+x) = m^*(A)$

(ii) A is measurable.

(iii) $\mathcal{H}(\mathcal{R})$ is a σ-ring and is the smallest hereditary σ-ring containing $\mathcal{R}$.

(iv) A triple $[X, \mathcal{S}, \mu]$ is called a measure space if $[X, \mathcal{S}]$ is a measurable space and μ is a measure on $\mathcal{S}$.

Example: $[\mathbb{R}, \mathcal{M}, m]$ and $[\mathbb{R}, \mathcal{B}, m]$ are measure spaces, where $\mathcal{B}$ denotes the Borel sets.

(V) A measurable simple function ϕ is one taking a finite number of non-negative values, each on a measurable set; so if $a_1, \dots, a_n$ are distinct values of ϕ, we have $\phi = \sum_{i=1}^{n} a_i \chi_{A_i}$, where $A_i = [x : \phi(x) = a_i]$. Then the integral of ϕ w.r.t μ (measure) is given by

$$\int \phi \, d\mu = \sum_{i=1}^{n} a_i \, \mu(A_i).$$

(vi) The total variation of a signed measure ν is $|\nu| = \nu^+ + \nu^-$, where $\nu = \nu^+ - \nu^-$ is the Jordan decomposition of ν.
Clearly $|\nu|$ is a measure on a measurable space $[X, S]$ and for each $E \in S$, $|\nu(E)| \leq |\nu|(E)$.

(vii) If μ, ν are signed measure on $[X, S]$ and $\nu(E) = 0$ whenever $|\mu|(E) = 0$, then ν is absolutely continuous w.r.t μ and we write $\nu << \mu$, where $[X, S]$ is a measurable space.

(viii) If X and Y are sets, their Cartesian product $X \times Y$ is the set of ordered pairs $[(x,y) : x \in X, y \in Y]$.
If X and Y are spaces, $X \times Y$ is the product space.

(ix) The Lebesgue outer measure of a set is given by $m^*(A) = \inf \sum l(I_n)$, where the infimum is taken over all finite or countable collections of intervals $[I_n]$ such that $A \subseteq \cup I_n$.

(x) <u>G_δ - Set</u> - A set is said to be G_δ if it is countable intersection of open sets.

2. Lebesgue measurable set -

The set E is Lebesgue measurable set if for each set A we have

$$m^*(A) = m^*(A \cap E) + m^*(A \cap E^c) \quad \text{——— (1)}$$

Now, we have to show that $\mathfrak{M}$ the class of Lebesgue measurable sets is a σ- algebra

From the definition of Lebesgue measurable set, $\mathbb{R} \in \mathfrak{M}$ and the symmetry in above definition between E and E^c implies that if $E \in \mathfrak{M}$ then $E^c \in \mathfrak{M}$. So it remains to be shown that if $\{E_j\}$ is a sequence of sets in $\mathfrak{M}$ then $E = \bigcup_{j=1}^{\infty} E_j \in \mathfrak{M}$.

Let A be an arbitrary set. By (1) (with E replaced by E_1) we have

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_1^c), \text{ we have}$$

again by (1) (with E replaced by E_1 & A by $A \cap E_1^c$)

$$m^*(A) = m^*(A \cap E_1) + m^*(A \cap E_2 \cap E_1^c) + m^*(A \cap E_1^c \cap E_2^c).$$

Continuing in this way we obtain, for $n \geq 2$,

$$m^*(A) = m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \left(\bigcap_{j<i} E_j^c\right)\right) + m^*\left(A \cap \left(\bigcap_{j=1}^{n} E_j^c\right)\right)$$

$$= m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) + m^*\left(A \cap \left(\bigcup_{j=1}^{n} E_j\right)^c\right)$$

$$\geq m^*(A \cap E_1) + \sum_{i=2}^{n} m^*\left(A \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) + m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right)$$

(Using the property that if $A \subseteq B$ then $m^*(A) \leq m^*(B)$)

Therefore

$$m^*(A) \geq m^*(A \cap E_1) + \sum_{i=2}^{\infty} m^*\left(A \cap E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) + m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right)$$

$$\geq m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)\right) + m^*\left(A \cap \left(\bigcup_{j=1}^{\infty} E_j\right)^c\right)$$

$$\geq m^*(A), \qquad \text{--- (2)}$$

Using the theorem, for any seq of sets $\{E_i\}$, $m^*\left(\bigcup_{i=1}^{\infty} E_i\right) \leq \sum_{i=1}^{\infty} m^*(E_i)$. and using the fact that for any n, $\bigcup_{i=1}^{n}\left(E_i \cap \left(\bigcup_{j<i} E_j\right)^c\right) = \bigcup_{i=1}^{n} E_i$

Hence we have equality throughout in (2) and we have shown that $\bigcup_{j=1}^{\infty} E_j$ is measurable. □

Now it is given that $F \in \mathfrak{M}$ & $m^*(F \Delta G) = 0$ we have to show that G is measurable.

we know that $m^*(F \Delta G) = 0 \Rightarrow F \Delta G$ is measurable. $F \Delta G = (F - G) \cup (G - F)$. so its subsets

$F-G$ & $G-F$ are measurable. So by above result, $F \cap G = F - (F-G)$ is measurable. So $G = (F \cap G) \cup (G-F)$ is measurable.

3. Proof - If E contains no set of negative ν-measure, then E is a positive set and $A = E$ gives the result. Otherwise there exists $n \in \mathbb{N}$ such that there exists $B \in \mathcal{S}$, $B \subseteq E$ and $\nu(B) < -1/n$. Let n_1 be the smallest such integer and E_1 a corresponding measurable subset of E with $\nu(E_1) < -1/n_1$. Let n_k be the smallest positive integer such that there is a measurable subset E_k of $E - \bigcup_{i=1}^{k-1} E_i$ with $\nu(E_k) < -1/n_k$. From the construction, $n_1 \leq n_2 \leq \cdots$ and we have a corresponding sequence $\{E_i\}$ of disjoint subsets of E. If the process stops, at n_m say, and $C = E - \bigcup_{i=1}^{m} E_i$, then C is a positive set, and $\nu(C) > 0$, for $\nu(C) = 0$ would imply that $\nu(E) = \sum_{i=1}^{m} \nu(E_i) < 0$. So C is the desired set. If the process does not stop, put $A = E - \bigcup_{k=1}^{\infty} E_k$; we wish to show that A is a positive set. We have

$$\nu(E) = \nu(A) + \nu\left(\bigcup_{k=1}^{\infty} E_k\right). \qquad \text{(1)}$$

But ν cannot take both the values $\infty, -\infty$, $\nu(E) > 0$ and $\nu\left(\bigcup_{k=1}^{\infty} E_k\right) = \sum_{k=1}^{\infty} \nu(E_k) < 0$, so the second term on the right-hand side of (1) is finite. So $\sum_{k=1}^{\infty} \nu(E_k) > -\infty$, hence $\sum_{k=1}^{\infty} 1/n_k < \infty$ and, in particular, $\lim_{k \to \infty} n_k = \infty$, and $n_k > 1$ for $k > k_0$, say. So let $B \in S$, $B \subseteq A$ and $k > k_0$. Then $B \subseteq E - \bigcup_{i=1}^{k} E_i$ so

$$v(B) \geq -\frac{1}{n_k - 1} \qquad \text{——— ②}$$

by the definition of n_k. But ② holds for all $k > k_0$, So letting $k \longrightarrow \infty$ we have $v(B) \geq 0$ and so A is a positive set. As before, $v(A) = 0$ would imply $v(E) < 0$, so $v(A) > 0$ as required.

④ Let A, B be a Hahn decomposition w.r.t μ, so that $\mu(E \cap A) = \mu^+(E)$, $-\mu(E \cap B) = \mu^-(E)$.
Now $v \ll \mu^+$ and μ^+ is σ-finite, so on applying Radon-Nikodym Theorem to μ^+ on A we get $v(E \cap A) = \int_{E \cap A} f_1 \, d\mu^+$ for an appropriate function f_1 on A, and Similarly $v(E \cap B) = \int_{E \cap B} f_1 \, d\mu^-$ for f_2 defined on B. So define $f = f_1$ on A, $f = -f_2$ on B. Then by the property of measurable functions, we get f is a measurable function on X, and $v(E) = \int_{A \cap E} f_1 \, d\mu^+ - \int_{B \cap E} (-f_2) \, d\mu^-$.

As v is a signed measure this will not be of the form $\infty - \infty$ so $v(E) = \int_E f \, d\mu$ is well defined. Any two such functions, from the construction, agree except on a set of zero μ^+ and μ^- measure, giving the result. $\square$

(5) (5) Let g be a finite valued left continuous monotonic (4) increasing function on $\mathbb{R}$ and for an interval $[a, b)$ define

$$\mu([a,b)) = g(b) - g(a)$$

If $a = b$, $\mu(\phi) = 0$

μ is non negative.

$\underline{I^{st}}$ Write $U_i = (a_i, b_i)$ and select intervals as follows Let $a \in U_{k_1}$, say. If $b_{k_1} \leq b$, let K_2 be such that $b_{k_1} \in U_{k_2}$ etc., by induction, the sequence ending when $b_{k_m} > b$. Renumbering the intervals, we have chosen $U_1, U_2, --- U_m$, where $a_{i+1} < b_i < b_{i+1}$, $i = 1, 2, --- m-1$, So

$$g(b) - g(a) \leq g(b_m) - g(a_1)$$

$$= g(b_1) - g(a_1) + \sum_{i=1}^{m-1} (g(b_{i+1}) - g(b_i))$$

$$\leq \sum_{i=1}^{m} \{g(b_i) - g(a_i)\}$$

But $m \leq n$ and $g(b) - g(a) \leq \sum_{i=1}^{n} \{g(b_i) - g(a_i)\}$ follows.

$\underline{II^{nd}}$ - (i) Suppose that $I = [a, b)$ and $E_i = [a_i, b_i)$, each i. We may suppose that $b > a$ and if $\varepsilon > 0$ we may chose c such that $0 < c < b-a$ and $g(b) - g(b-c) < \varepsilon$. Also, for each i, choose ξ_i such that $\xi_i < a_i$ and $g(a_i) - g(\xi_i) < \varepsilon/2^i$. Write $F = [a, b-c]$ and $U_i = (\xi_i, b_i)$. So $F \subset \bigcup_{i=1}^{\infty} U_i$ and by Heine Borel Theorem, $F \subset \bigcup_{i=1}^{n} U_i$. Then by Theorem 2

$$g(b-c) - g(a) \leq \sum^{n} (g(b_i) - g(\xi_i))$$

$$< \sum_{i=1}^{n} \left(g(b_i) - g(a_i) + \varepsilon/2^i \right)$$

$$< \sum_{i=1}^{\infty} \left(g(b_i) - g(a_i) \right) + \varepsilon .$$

So $g(b) - g(a) - \varepsilon < \sum_{i=1}^{\infty} \mu(E_i) + \varepsilon$.

But ε is arbitrary and result follows.

(ii) As we know, if E_i, $i = 1, \ldots, n$, are disjoint intervals such that $\bigcup_{i=1}^{n} E_i \subseteq I$, where I is an interval, then $\sum_{i=1}^{n} \mu(E_i) \leq \mu(I)$. From this we obtain $\sum_{i=1}^{\infty} \mu(E_i) \leq \mu(I)$.

But then (i) gives the result.

(6) <u>Riesz Representation Theorem</u> : Let G be a bounded linear functional on $L^p(X, \mu)$. Then there exists a unique element g of $L^q(X, \mu)$ such that

$$G(f) = \int f g \, d\mu \qquad \text{for each } f \in L^p \quad \text{— (1)}$$

where p, q are conjugate indices. Also

$$\|G\| = \|g\|_q . \qquad \text{— (2)}$$

<u>Proof</u> - Let g and g' have the desired property and let E be any set of finite measure, so that $\chi_E \in L^p$.

Then $\int_E (g - g') \, d\mu = \int \chi_E (g - g') \, d\mu = 0$.

So $g = g'$ a.e., since the set $[x : g(x) \neq g'(x)]$ has σ-finite measure. So the uniqueness is proved.

Let G on $L^p(\mu)$ defined by $G(f) = \int fg\,d\mu$ for a fixed $g \in L^q(\mu)$, p and q being conjugate indices with $p \geq 1$ and with $q = \infty$ in the case where $p = 1$. Then G is a bounded linear functional and $\|G\| \leq \|g\|_q$. ⑤

(*)

If $\|G\| = 0$ then $G(f) = 0$ for all f, so $g \equiv 0$ satisfies ① and ②. So suppose $\|G\| > 0$. Suppose first that $\mu(X) < \infty$. For each $E \in \mathcal{S}$ define $\lambda(E) = G(\chi_E)$

Firstly we show that λ is a signed measure.

clearly $\lambda(\phi) = 0$. Since $\chi_{A\cup B} = \chi_A + \chi_B$ for disjoint sets A, B, λ is finitely additive. Let $E = \bigcup_{i=1}^{\infty} E_i$ and let $A_n = \bigcup_{i=1}^{n} E_i$. We have $\|\chi_{A_n} - \chi_E\|_p = (\mu(E - A_n))^{1/p} \to 0$ as $n \to \infty$.

Since G is continuous, we have $\lambda(A_n) \longrightarrow \lambda(E)$, so λ is countably additive. Since G takes only finite values, λ is a signed measure. Also if $\mu(E) = 0$, then $\|\chi_E\|_p = 0$ so $\lambda(E) = 0$, i.e. $\lambda \ll \mu$. So by Corollary of Radon - Nikodym Theorem, i.e R.N. theorem can be extended to the case where ν is a σ-finite signed measure, there exists $g \in L^1(\mu)$ s.t. for each $E \in \mathcal{S}$

$$G(\chi_E) = \int_E g\,d\mu = \int \chi_E\, g\,d\mu .$$

We now dispense with the signed measure λ and show that g has the required properties.

By linearity we have $G(\phi) = \int \phi g\,d\mu$ for any measurable simple function ϕ. But each function $f \in L^\infty(\mu)$ is the uniform limit a.e of a seq $\{\psi_n\}$ where each ψ_n is the

difference of measurable simple functions, so $\|f-\psi_n\|_p \to 0$.
So, by the continuity of G,

$$G(f) = \int f g \, d\mu \quad \text{for each } f \in L^\infty(\mu). \text{ --- } (3)$$

Now we show that $\|G\| = \|g\|_q$. Let the function α on X ~~Again~~ be defined by: $\alpha = 1$ where $g > 0$, $\alpha = -1$ where $g \leq 0$,
So α is measurable and $\alpha g = |g|$. Let $E_n = \{x : |g(x)| \leq n\}$
and put $f = \alpha \chi_{E_n} |g|^{q-1}$ where p, q are conjugate indices. Then $|f|^p = |g|^q$ on E_n, $f \in L^\infty(\mu)$ and by (3)

$$\int_{E_n} |g|^q \, d\mu = \int f g \, d\mu = G(f) \leq \|G\| \, \|f\|_p$$
$$= \|G\| \left(\int_{E_n} |g|^q \, d\mu \right)^{1/p} \text{ --- } (4)$$

So we get $\int \chi_{E_n} |g|^q \, d\mu \leq \|G\|^q$ --- (5)

For this is obvious if $\left(\int_{E_n} |g|^q \, d\mu \right)^{1/p} = 0$; otherwise divide (4) across by this factor and raise to the power q. Since $\chi_{E_n} \uparrow 1$, (5) & R. H. Th^m, give $\|g\|_q \leq \|G\|$, and, in particular, $g \in L^q(\mu)$.
So by (*) $\|g\|_q = \|G\|$,

So (1) holds for $f \in L^\infty(X, \mu)$. But the bounded functions are dense in L^p. For it is sufficient to show that every non-negative function $f \in L^p$ is the limit, in the mean of order p, of a sequence $\{f_n\}$ of bounded functions. Put $f_n = \min(f, n)$. Then $0 \leq (f - f_n)^p \leq f^p$ and $f - f_n \to 0$ a.e. So by Lebesgue's Dominated Convergence

Theorem; $\|f-f_n\|_p \longrightarrow 0$. Then by the Continuity of G, $G(f_n) \to G(f)$. Also by Holder's inequality, $\int f_n g \, d\mu \longrightarrow \int f g \, d\mu$. So $G(f) = \int f g \, d\mu$, proving the result of the theorem of finite measure spaces.

We now extend the result to the case when $X = \bigcup_{i=1}^{\infty} X_i$, where the X_i are disjoint measurable sets of finite μ-measure. Any function f_i on X_i, measurable w.r.t the σ-algebra of sets $E \cap X_i$, $E \in \mathcal{S}$ can be extended to f on X by putting $f_i = 0$ on X_i^c. Then G has the restriction G_i on $L(X, \mu)$ where $G_i(f_i) = G(f)$, and we have $\|G_i\| \le \|G\|$. By the first part, $G_i(f_i) = G(\chi_{X_i} f) = \int_{X_i} f g_i \, d\mu$ for each $f \in L^p(X, \mu)$, for each i, and for a suitable $g_i \in L^q(X_i, \mu)$. Extend g_i to X by putting $g_i = 0$ on X_i^c and write $g = \sum g_i$. By linearity, If $Y_n = \bigcup_{i=1}^{n} X_i$,

$$G(\chi_{Y_n} f) = \int_{Y_n} f(g_1 + g_2 + \cdots + g_n) \, d\mu, \quad \forall f \in L^p(X, \mu).$$

As in the first part, since $\mu(Y_n) < \infty$, we have

$$\|g_1 + g_2 + \cdots + g_n\| \le \|G\| \text{ for each } n. \text{ So}$$

$$(\|g\|_q)^q = \int |\Sigma g_i|^q \, d\mu = \int \lim \left|\sum_{i=1}^{n} g_i\right|^q d\mu$$

$$\le \liminf \int \left|\sum_{i=1}^{n} g_i\right| d\mu \le \|G\|^q$$

by Fatou's Lemma, giving $\|g\|_q = \|G\|$ by ⊛. Also $\chi_{Y_n} f \longrightarrow f$ in the mean of order p so $G(\chi_{Y_n} f) \longrightarrow G(f)$. But $\sum^n g_i \longrightarrow g$ in the mean of order q, so by Holder's inequality $\int \chi_{Y_n} f \sum_{i=1}^{n} g_i \, d\mu \Rightarrow \int f g \, d\mu$.

Now consider the general case where μ need not be σ-finite. We show that there exists a set $X_0 \in \mathcal{S}$ which is of σ-finite measure, that is, X_0 is the union of a sequence of sets of finite measure and such that if $f = 0$ on X_0 then $G(f) = 0$. Let $\{f_n\}$ be such that $\|f_n\|_p = 1$ and $G(f_n) \geq \|G\| (1 - 1/n)$. We know that $X_0 = \bigcup_{n=1}^{\infty} [x : f_n(x) \neq 0]$ has σ-finite measure. Let $E \in \mathcal{S}$ with $E \subseteq X_0^c$, then $\|f_n + t\chi_E\|_p = (1 + t^p \mu(E))^{1/p}$ for $t \geq 0$. Also

$$G(f_n) - G(\pm t\chi_E) \leq |G(f \pm t\chi_E)|$$
$$\leq \|G\| (1 + t^p \mu(E))^{1/p}$$

and it follows that

$$|G(t\chi_E)| \leq \|G\| [(1 + t^p \mu(E))^{1/p} - 1 + n^{-1}]$$

for every n. Let $n \to \infty$ and then divide by $t (> 0)$ to get

$$|G(\chi_E)| \leq \|G\| \frac{(1 + t^p \mu(E))^{1/p} - 1}{t}.$$

Since $p > 1$ we may apply L' Hospital's rule as $t \to 0$ to get $G(\chi_E) = 0$. So G vanishes for simple functions and hence for measurable functions which equal zero on X_0. So by the proof for the σ-finite case we can find $g \in L^q(X_0)$ such that

$$G(\chi_{X_0} f) = \int_{X_0} f g \, d\mu.$$

Define g to be zero on X_0^c to get the required functional of the theorem.

$\square$

7 Hahn decomposition theorem

Let v be a signed measure on $[X, \mathcal{S}]$. Then there exists a positive set A and a negative set B such that $A \cup B = X$, $A \cap B = \phi$. The pair A, B is said to be Hahn decomposition of X w.r.t v. It is unique to the extent that if A_1, B_1 and A_2, B_2 are Hahn decomposition of X w.r.t v, then $A_1 \Delta A_2$ is a v-null set.

Proof:- We may suppose that $v < \infty$ on $\mathcal{S}$, for otherwise we consider $-v$, the result of the theorem for $-v$ implying the result for v. Let $\lambda = \sup [v(C) : C \text{ is a positive set}]$, so $\lambda \geq 0$. Let $\{A_i\}$ be a sequence of positive set, (and from the definition of λ) such that $\lambda = \lim v(A_i)$. A countable union of sets positive w.r.t signed measure v is a positive set, therefore $A = \bigcup_{i=1}^{\infty} A_i$ is a positive set, and from the definition of λ, $\lambda \geq v(A)$. But $A - A_i \subseteq A$ and hence is a positive set. So for each i,

$$v(A) = v(A_i) + v(A - A_i) \geq v(A_i)$$

So $v(A) \geq \lim v(A_i) = \lambda$ and hence $v(A) = \lambda$, that is, the value of λ is achieved on a positive set. Write $B = A^c$. Then if B contains a set D of positive ~~set~~ v-measure, we have $0 < v(D) < \infty$. [Let v (signed measure) on $[X, \mathcal{S}]$. Let $E \in \mathcal{S}$ and $v(E) > 0$. Then there exists A, a set positive w.r.t v, s.t $A \subseteq E$ & $v(A) > 0$] ✡
So by ✡, D contains a positive set E such that $0 < v(E) < \infty$. But then $v(A \cup E) = v(A) + v(E) > \lambda$, contradicting the definition of λ. So $v(D) \leq 0$ and B is a negative set

and A, B form a Hahn decomposition.

For the last part note that $A_1 - A_2 = A_1 \cap B_2$ and hence is a positive and negative set and so a null set. Similarly $A_2 - A_1$ is a null set, and so $A_1 \Delta A_2$ is null.

$\square$

(8) Each set $E \in \mathcal{R}$ can be written as $E = \bigcup_{i=1}^{n} E_i$ where the E_i are disjoint intervals. Define $\bar{\mu}(E) = \sum_{i=1}^{n} \mu(E_i)$.

This defines $\bar{\mu}$ uniquely on $\mathcal{R}$ since if $E = \bigcup_{j=1}^{m} F_j$ is another decomposition of E into disjoint intervals, then $E = \bigcup_{i,j} (E_i \cap F_j)$, the intervals $E_i \cap F_j$ are disjoint and

$$\bar{\mu}(E) = \sum_{i=1}^{n} \mu(E_i) = \sum_{i=1}^{n}\sum_{j=1}^{m} \mu(E_i \cap F_j)$$

$$= \sum_{j=1}^{m}\sum_{i=1}^{n} \mu(E_i \cap F_j) = \sum_{j=1}^{m} \mu(F_j),$$

using the additivity of μ ~~on given by~~. So μ and $\bar{\mu}$ are equal for intervals; also $\bar{\mu}$ is clearly finitely additive.

Let $\{E_i\}$ be a sequence of disjoint sets of $\mathcal{R}$ such that $E = \bigcup_{i=1}^{\infty} E_i \in \mathcal{R}$. Then, for each i, E_i is a finite union of disjoint intervals

$$E_i = \bigcup_{j=1}^{m(i)} E_{ij},$$

so $\bar{\mu}(E_i) = \sum_{j=1}^{m(i)} \mu(E_{ij})$. If E is an interval, then

$$\bar{\mu}(E) = \mu(E) = \sum_{i=1}^{\infty}\sum_{j=1}^{m(i)} \mu(E_{ij}) = \sum_{i=1}^{\infty} \bar{\mu}(E_i)$$

(1)

as the intervals E_{ij} are disjoint. In general, we can write $E = \bigcup_{K=1}^{m} F_K$ where the F_K are disjoint intervals.

Then, as $\bar{\mu}$ is finitely additive

$$\bar{\mu}(E) = \sum_{K=1}^{m} \bar{\mu}(F_K) = \sum_{K=1}^{m} \sum_{i=1}^{\infty} \bar{\mu}(F_K \cap E_i) \text{ by } ①.$$

So $\bar{\mu}(E) = \sum_{i=1}^{\infty} \sum_{K=1}^{m} \bar{\mu}(F_K \cap E_i) = \sum_{i=1}^{\infty} \bar{\mu}(E_i)$. So $\bar{\mu}$ is countably additive. Since $\bar{\mu}(\phi) = \mu(\phi) = 0$, $\bar{\mu}$ is a measure.

Clearly, any measure on $\mathcal{R}$ which extends μ must, from the definition of $\bar{\mu}$, equal $\bar{\mu}$ on each set $\mathcal{R}$.

So the extension is unique.

$\square$

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